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1. Problems 1.1.9, 1.1.10, Pg. I-286 of the class notes. What is the connection with this proof and the Contraction Mapping Theorem?

   (1.1.9+extra ?) Let (M, d) be a metric space, let $\{x_n\}$ be a sequence in M, and suppose that there are numbers k, $0 < k < 1$, and $C > 0$ such that $d(x_{n+1}, x_n) \leq Ck^n$ for all $n \geq 1$. Show that the sequence is Cauchy. What is the connection with this proof and the Contraction Mapping Theorem?

   Solution. First, notice that, given any $\epsilon > 0$, I can find find some N_ϵ s.t.

$$\frac{k^{N_\epsilon}}{1-k} < \frac{\epsilon}{C}$$

   for some $C > 0$ (the same C as in the problem statement).

   Now, given some $m > n > N_\epsilon$, let's look at the quantity $d(x_m, x_n)$:

$$\begin{aligned} d(x_m, x_n) &\leq d(x_m, x_{m-1}) + \cdots + d(x_{n+1}, x_n) \\ &\leq C(k^{m-1} + k^{m-2} + \cdots + k^n) \\ &= Ck^n(k^{m-n-1} + k^{m-n-2} + \cdots + 1) \\ &< Ck^{N_\epsilon}(k^{m-n-1} + k^{m-n-2} + \cdots + 1) \end{aligned}$$

   Since $k \in (0, 1)$, $k^j < k^l$ for $j > l$. Notice now that the term in parentheses aboves looks like a geometric series which has sum $\frac{1}{1-k}$, therefore

$$\begin{aligned} d(x_m, x_n) &< Ck^{N_\epsilon}\frac{1}{1-k} \\ &< C\frac{\epsilon}{C} \\ &= \epsilon \end{aligned}$$

   So, the sequence $\{x_n\}$ is Cauchy. Now address the connection of this proof with the Contraction Mapping Theorem (CMT).

   Suppose $T : M \to M$ is a contraction, where $T(x_n) = x_{n+1}$. Using the CMT, for $\alpha \in (0, 1)$,

$$\begin{aligned} d(x_{n+1}, x_n) &= d(T(x_n), T(x_{n-1}) \\ &\leq \alpha d(x_n, x_{n-1}). \end{aligned}$$

   This sequence is Cauchy (follow arguments from 1.1.9 solution above). So, the sequence has a limit. Let $y = \lim_{n\to\infty} x_n$,

$$\begin{aligned} T(y) &= T(\lim_{n\to\infty} x_n) \\ &= \lim_{n\to\infty} T(x_n) \\ &= \lim_{n\to\infty} x_{n+1} \\ &= y. \end{aligned}$$

So, y is a fixed point of T. Now, show that it is the unique fixed point.

Suppose $y \neq z$ and $T(y) = y$ and $T(z) = z$, then

$$\begin{aligned} d(y,z) &= d(T(y),T(z)) \\ &\leq \alpha d(y,z) \\ &< d(y,z) \end{aligned}$$

which is clearly a contradiction. So the fixed point y is unique.

(1.1.10) Define a sequence of real numbers by $x_1 = 2$,

$$x_{n+1} = \frac{x_n}{2} + \frac{1}{x_n}.$$

(This is the Newton-Raphson method for finding $\sqrt{2}$.) Prove that $\{x_n\}$ is a Cauchy sequence, *without* using the fact that $\lim x_n = \sqrt{2}$. (Hint: use the preceding exercise.)

Proof. Spot check the first couple differences,

$$\begin{aligned} |x_2 - x_1| &= \frac{1}{2} \leq \left(\frac{1}{2}\right)^1 \\ |x_3 - x_2| &= \frac{1}{12} \leq \left(\frac{1}{2}\right)^2 \\ |x_4 - x_3| &= \frac{1}{408} \leq \left(\frac{1}{2}\right)^3 \end{aligned}$$

Follow the argument inductively and we arrive at the relationship

$$|x_{n+1} - x_n| \leq \left(\frac{1}{2}\right)^n.$$

Now, we can apply the result from the proof from (1.1.9) with $k = \frac{1}{2}$ and $C = 1$. Therefore the sequence is Cauchy.

2. Problem 1.1.16, Pg. II-289

   (1.1.16) Use the same sequence f_n and modify the last argument to prove that $\mathcal{C}([0,1])$ is not complete in the L^2-metric.

   Proof. Start with the same definitions for f_n and ψ as in the book:

$$f_n = \begin{cases} -1 & 0 \leq t \leq \frac{1}{2} - \frac{1}{n} \\ nt - \frac{n}{2} & \frac{1}{2} - \frac{1}{n} \leq t \leq \frac{1}{2} + \frac{1}{n} \\ +1 & \frac{1}{2} + \frac{1}{n} \leq t \leq 1 \end{cases}$$

which has pointwise limit

$$\psi = \begin{cases} -1 & 0 \le t < \frac{1}{2} \\ 0 & t = \frac{1}{2} \\ +1 & \frac{1}{2} < t \le 1 \end{cases}$$

We want to show that no continuous function can be the L^2 limit of f_n. Suppose the opposite were true. Start with Minköwski inequality, assume $f \in \mathcal{C}([0,1])$,

$$||f - \psi||_2 \le ||f - f_n||_2 + ||f_n - \psi||_2.$$

By assumption $||f - f_n||_2 \to 0$. Now evaluate $||f_n - \psi||_2$:

$$\begin{aligned} ||f_n - \psi||_2 &= \sqrt{\int_0^{1/2-1/n} \left(-1-(-1)\right)^2 dt} + \sqrt{\int_{1/2-1/n}^{1/2} \left(nt - \frac{n}{2} + 1\right)^2 dt} \\ &+ \sqrt{\int_{1/2}^{1/2+1/n} \left(nt - \frac{n}{2} - 1\right)^2 dt} + \sqrt{\int_{1/2+1/n}^{1} \left(1-1\right)^2 dt} \\ &= I + II + III + IV. \end{aligned}$$

Clearly I, IV are 0, since the integrand is 0 in both cases. II, III are dealt with via handy u substitutions which leave the integrals

$$\begin{aligned} ||f_n - \psi||_2 &= \sqrt{\frac{1}{n}\int_0^1 u^2 du} + \sqrt{\frac{1}{n}\int_{-1}^0 u^2 du} \\ &= 2\sqrt{\frac{1}{3n}} \end{aligned}$$

which can be made arbitrarily small as $n \to \infty$. So, $||f_n - \psi||_2 = 0$. Therefore, $||f - \psi||_2 = 0$. This, however, is a contradiction.

Since f is continuous, there must be a $t_0 \in [0, 1/2)$ s.t. $f(t_0) \neq -1$ or a $t_0 \in (1/2, 1]$ s.t. $f(t_0) \neq +1$. Suppose the latter, WLOG. By continuity of f, $|f(t) - 1|^2 > \eta > 0$, for some η, in some neighborhood of t_0. Therefore, the integral of $|f(t) - \psi(t)|^2 > 0$, and therefore the square-root of this integral is positive. So, $||f - \psi|| \neq 0$.

$\therefore$ $\mathcal{C}([0,1])$ is not complete in the L^2-metric.

3. Problem 1.1.39, Pg. II-297

   (1.1.39) Prove: if (M, d) is a complete metric space, and if a set A and its complement are both dense, then they cannot both be F_σ sets.

   Proof. First, a few definitions/theorems to be used.

   Definition 1. A set is said to be F_σ if it is a countable union of closed sets.

   Theorem 1. Baire Category Theorem. *In a complete metric space, every residual set is dense.*

Definition 2. A residual set is defined to be a countable intersection of open dense sets.

Now, on to the proof.

Suppose A, A^c are both dense in M and they are both F_σ sets. First work with A. For some indexing set I,

$$A = \bigcup_{i \in I} F_i$$

for closed sets F_i. Taking the complement of A as defined, we get

$$A^c = \bigcap_{i \in I} F_i^c$$

which is dense in M by assumption. Therefore all of the F_i^c are dense, and A^c is a residual set.

If we follow through the argument of the last paragraph, considering instead A^c as being F_σ instead of A, A is also a residual set. Now, we work with Theorem 1 above: We know that the metric space (M, d) is complete, therefore every residual set is dense. But, to show a contradiction, consider

$$A \cap A^c = \emptyset.$$

This is clearly a residual set, since it is a countable intersection of open dense sets, but $\emptyset$ is not dense. This contradicts the assumption of (M, d) being complete.

$\therefore$ if a set A and its complement are both dense, then they cannot both be F_σ sets.

4. Problem 1.1.42, Pg. II-297

Let (M, d) be a metric space. Let $\mathcal{C}(M)$ be the linear space of continuous real-valued functions on M for which the norm

$$||f||_\infty = \sup_{x \in M} |f(x)|$$

is finite. Prove that $\mathcal{C}(M)$ is complete in the metric derived from this norm.

Proof. From definition of functions in $\mathcal{C}(M)$, we have that

$$\begin{aligned} |f_n(x) - f_m(x)| &\leq ||f_n - f_m||_\infty \\ &= d(f_n, f_m). \end{aligned}$$

The sequence $\{f_n(x)\}$ is Cauchy in $\mathcal{C}(M)$, since M is complete, and therefore is convergent. Define $f(x)$ to be

$$f(x) = \lim_{n \to \infty} f_n(x).$$

Now, I need to show first that $f \in \mathcal{C}(M)$.

Since $\{f_n\}$ is Cauchy, then $\exists N_1$ s.t. for every $n, m \geq N_1$

$$d(f_n, f_m) < 1,$$

and so $|f_n(x) - f_{N_1}(x)| < 1, \forall x \in M, n \geq N_1$. Since $f_{N_1} \in \mathcal{C}(M)$, then $\exists \delta$ s.t. $|f_{N_1}(x)|\delta$, $\forall x \in M$. Then

$$\begin{aligned} |f_n(x)| &\leq |f_n(x) - f_{N_1}(x)| + |f_{N_1}(x)| \\ &\leq 1 + \delta. \end{aligned}$$

This result holds for all n, therefore

$$\begin{aligned} |f(x)| &= \lim_{n\to\infty} |f_n(x)| \\ &\leq 1 + \delta \;\; \forall x \in M \end{aligned}$$

so $f \in \mathcal{C}(M)$.

So, the candidate limit is in the space $\mathcal{C}(M)$, but is it the actual limit, i.e. $\lim_{n\to\infty} d(f_n, f) = 0$. Given $\epsilon > 0$ and the fact that $\{f_n\}$ is Cauchy. So, $\exists N_\epsilon$ s.t. for $n, m \geq N_\epsilon$, we have that $d(f_n, f_m) < \epsilon, \forall x \in M$. Therefore,

$$\lim_{m\to\infty} |f_n(x) - f_m(x)| = |f_n(x) - f(x)| < \epsilon$$

since the inequality holds for all m.

$\therefore \mathcal{C}(M)$ is complete in the metric derived from this norm.

5. Problem 1.1.47, Pg. II-297

(1.1.47) Let $\mathcal{C}^k([0,1])$ be the linear space of functions defined on $[0,1]$ whose first k derivatives exist and are continuous. Equip it with the norm

$$||f|| = \sup_{y\in[0,1]} |f(y)| + \sup_{y\in[0,1]} |f'(y)| + \cdots + \sup_{y\in[0,1]} |f^{(k)}(y)|.$$

Prove completeness.

Proof. Try proof by induction. We know that $\mathcal{C}^1([0,1])$ is complete from an example in the class notes. Now, assume $\mathcal{C}^m([0,1])$ with norm

$$||f||_{\mathcal{C}^m} = \sum_{i=0}^{m} \sup_{y\in[0,1]} |f^{(i)}(y)|$$

is complete. Now consider $\mathcal{C}^{m+1}([0,1])$ with norm

$$||f||_{\mathcal{C}^{m+1}} = \sum_{i=0}^{m} \sup_{y\in[0,1]} |f^{(i)}(y)| + \sup_{y\in[0,1]} |f^{m+1}(y)|.$$

We know that the sequences $\{f_n\}, \{f_n^{(1)}\}, \ldots, \{f_n^{(m)}\}$ are Cauchy wrt $C^m([0,1])$ and therefore have limits in $C^m([0,1])$. Now given this information, I need to show that $f_n^{(m+1)} \to \xi, \xi \in C^{m+1}([0,1])$. By the Fundamental Theorem of Calculus,

$$f_n^{(m)}(x) - f_n^{(m)}(0) = \int_0^x f_n^{(m+1)}(t)dt$$

But, we know that the limit exists for the term on the LHS of the above equation, that is

$$f_n^{(m)}(x) - f_n^{(m)}(0) \to f^{(m)}(x) - f^{(m)}(0)$$

It seems clear given this information that the limit $\xi(t)$ exists and is equal to $f^{(m+1)}(t)$, since $f \in \mathcal{C}^{m+1}([0,1])$ by assumption. Now, I just need to show that
$||f_n^{(m+1)} - f^{(m+1)}||_\infty \to 0$. Consider

$$\begin{aligned}\left|\int_0^x f_n^{(m+1)}(t) - f^{(m+1)}(t)dt\right| &= |f_n^{(m)}(x) - f_n^{(m)}(0) - (f^{(m)}(x) - f^{(m)}(0))| \\ &\to 0 \\ \implies f_n^{(m+1)}(t) &\to f^{(m+1)}(t) \ \ \forall t \in [0,1] \\ \implies ||f_n^{(m+1)} - f^{(m+1)}||_\infty &\to 0\end{aligned}$$

Therefore $\{f_n^{(m+1)}\}$ is Cauchy in $\mathcal{C}^{m+1}([0,1])$ and converges to a limit in $\mathcal{C}^{m+1}([0,1])$. Therefore $\mathcal{C}^{m+1}([0,1])$ is complete.

$\therefore$ is complete for any choice of k.

6. Problem 1.1.48, Pg. II-298

(1.1.48) Let $\mathbf{A}$ be a real $n \times n$ matrix, and suppose that $||\mathbf{A}|| < 1$ for some natural matrix norm $||\cdot||$. Use the Contraction Mapping Theorem to prove that the equation $\mathbf{x} + \mathbf{A}\mathbf{x} = \mathbf{b}$ has a unique solution $\mathbf{x}$ for every given $\mathbf{b} \in \mathbb{R}^n$.

Proof. The CMT applies directly here. First, notice that $\mathbb{R}^n$ is complete. Change the form of the equation to

$$\mathbf{x} = \mathbf{b} - \mathbf{A}\mathbf{x} = f(\mathbf{x})$$

Now, for two points $\mathbf{x}, \mathbf{z}$,

$$\begin{aligned} d(f(\mathbf{x}), f(\mathbf{z})) &= d(\mathbf{b} - \mathbf{A}\mathbf{x}, \mathbf{b} - \mathbf{A}\mathbf{x}) \\ &\leq ||(\mathbf{b} - \mathbf{A}\mathbf{x}) - (\mathbf{b} - \mathbf{A}\mathbf{z})|| \\ &= ||\mathbf{A}(\mathbf{z} - \mathbf{x})|| \\ &\leq ||\mathbf{A}||\ ||\mathbf{z} - \mathbf{x}||\end{aligned}$$

Now, the CMT states that given the above condition (since $||\mathbf{A}|| < 1$) a unique $\mathbf{y}$ exists satisfying

$$\mathbf{y} + \mathbf{A}\mathbf{y} = \mathbf{b}.$$